Seminar Lecture - Set Theory

1. **Definition:** A **set** is a well-defined collection of objects meaning that it is possible to determine if an object belongs to the set or not, without prejudice.

2. The **empty set**, denoted $\emptyset$, is defined as the set containing no elements.

3. If A denotes a set and x denotes an object, or **element**, belonging to A, we write $x \in A$. If x does not belong to A, we write $x \notin A$.

4. The **empty set**, denoted $\emptyset$, is defined as the set containing no elements.

5. **Definition:** A **universal set** is a set containing all of the objects under consideration and will be denoted X. (There are fundamental issues with making such a set 'too big.' - see Russel's Paradox.)

6. If A and B are two sets, we say 'A is **equal** to B' and write $A = B$ if and only if A and B contain exactly the same elements. In symbols:

$$A = B \iff (\forall x : x \in A \iff x \in B)$$

   If every element of A is an element of B, then we say 'A is a **subset** of B' (B is a **superset** of A') and write $A \subseteq B$ ('$B \supseteq A$). In symbols:

$$A \subseteq B \iff (\forall x : x \in A \implies x \in B)$$

7. Hence, when speaking of sets, A, and B, we assume they are both subsets of a universal set, X.

8. **Definitions:** Let A and B be subsets of a universal set, X.

   (a) $\tilde{A} = \{x : x \notin A\}$ is called the *complement* of A.
   (b) $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$ is called the *intersection* of A with B.
   (c) If $A \cap B = \emptyset$, we say A and B are *disjoint* sets.
   (d) $A \cup B = \{x : (x \in A) \vee (x \in B)\}$ is called the *union* of A with B.
   (e) If A and B are known to be disjoint, we write $A \uplus B$ to signify $A \cup B$.

9. **Definition:** Given a set X, the *power set* of X, $\mathcal{P}(X)$ is: $\mathcal{P}(X) = \{A : A \subseteq X\}$. (i.e., $\mathcal{P}(X)$ is the set of all subsets of X.)

10. **VENN DIAGRAMS:** A way to 'visualize' sets.

11. **Theorem:** 'The Set Calculus': Let A, B, and C be subsets of a universal set, X.

   (a) $\tilde{\tilde{A}} = A$ (Idempotency of $\tilde{}$)
   (b) $A \cap B = B \cap A$ (Commutativity of $\cap$)
   (c) $A \cup B = B \cup A$ (Commutativity of $\cup$)
   (d) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity of $\cap$)
   (e) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity of $\cup$)
   (f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law: $\cap$ over $\cup$)
   (g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law: $\cup$ over $\cap$)
   (h) $\left.\begin{array}{l} A \cap \tilde{A} = \emptyset \\ A \cup \tilde{A} = X \end{array}\right\} \quad A \dot{\cup} \tilde{A} = X$
   (i) $\left.\begin{array}{l} \widetilde{A \cap B} = \tilde{A} \cup \tilde{B} \\ \widetilde{A \cup B} = \tilde{A} \cap \tilde{B} \end{array}\right\}$ DeMorgan's Laws
   (j) $A \cap X = A$ (Identity of $\cap$)
   (k) $A \cup \emptyset = A$ (Identity of $\cup$)
   (l) $A \cap \emptyset = \emptyset$ (Absorption of $\emptyset$ with $\cap$)
   (m) $A \cup X = X$ (Absorption of X with $\cup$)
   (n) $\tilde{X} = \emptyset$
   (o) $\tilde{\emptyset} = X$

12. **Theorem:** Properties of $\subseteq$: Let A, B, and C be subsets of a universal set, X.

   (a) $\emptyset \subseteq A$
   (b) $A \subseteq A$ (Reflexive Property)
   (c) $A = B \iff (A \subseteq B) \wedge (B \subseteq A)$
   (d) $(A \subseteq B) \wedge (B \subseteq C) \implies A \subseteq C$ (Transitive Property)
   (e) $A \cap B \subseteq A$
   (f) $A \cap B = A \iff A \subseteq B$
   (g) $A \subseteq A \cup B$
   (h) $A = A \cup B \iff B \subseteq A$
   (i) $A \subseteq B \implies A \cap C \subseteq B \cap C$
   (j) $A \subseteq B \implies A \cup C \subseteq B \cup C$

13. **Definition:** $A - B = \{x : x \in A \wedge x \notin B\}$ is called the *set theoretic difference* of A and B.

14. **Theorem:** Properties of $-$: Let A and B be subsets of a universal set, X:

   (a) $A - B = A \cap \tilde{B}$
   (b) $X - A = \tilde{A}$
   (c) $A - \emptyset = A$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$
   (e) $A - (B \cup C) = (A - B) \cap (A - C)$

Seminar Lecture - Families of Sets

1. A *family of sets* is a set of sets. We usually denote families of sets by calligraphic letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc.

2. **Example:** Given a set, X, $\mathcal{P}(X)$ is a family of sets.

3. $\displaystyle\bigcap_{A\in\mathcal{A}} A = \{x : \forall A \in \mathcal{A}, x \in A\}$; i.e., $\displaystyle x \in \bigcap_{A\in\mathcal{A}} A \iff \forall A \in \mathcal{A}, x \in A$

4. $\displaystyle\bigcup_{A\in\mathcal{A}} A = \{x : \exists A \in \mathcal{A} \ni x \in A\}$; i.e., $\displaystyle x \in \bigcup_{A\in\mathcal{A}} A \iff \exists A \in \mathcal{A} \ni x \in A$

5. Oftentimes, we *index* the family of sets. For example, $A_n = [-n, n]$ for $n \in \mathbb{N}$. In this case...

   - $\displaystyle\bigcap_{A\in\mathcal{A}} A = \bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty}[-n, n] = [-1, 1]$
   - $\displaystyle\bigcup_{A\in\mathcal{A}} A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}[-n, n] = (-\infty, \infty)$

6. In the previous example, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. These sets are called 'nested.'

7. In general, by an *indexed family of sets*, we mean a family of sets, $\mathcal{A}$ along with a function, $f : \Delta \twoheadrightarrow \mathcal{A}$. If $\alpha \in \Delta$, we write $f(\alpha) = A_\alpha$, and write the family: $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$

8. If we have an indexed family of sets...

   - $\displaystyle\bigcap_{A\in\mathcal{A}} A = \bigcap_{\alpha\in\Delta} A_\alpha$
   - $\displaystyle\bigcup_{A\in\mathcal{A}} A = \bigcup_{\alpha\in\Delta} A_\alpha$

9. **Theorem:** Provided $\Delta \neq \emptyset$:

   (a) $\displaystyle B \cap \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cap A_\alpha)$

   (b) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$

   (c) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

   (d) $\displaystyle B \cup \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cup A_\alpha)$

   (e) $$\left.\begin{array}{l} \displaystyle\widetilde{\bigcap_{\alpha\in\Delta} A_\alpha} = \bigcup_{\alpha\in\Delta} \tilde{A}_\alpha \\[2em] \displaystyle\widetilde{\bigcup_{\alpha\in\Delta} A_\alpha} = \bigcap_{\alpha\in\Delta} \tilde{A}_\alpha \end{array}\right\} \text{DeMorgan's Laws}$$